Lecture 3: Stacks are generalizations of schemes, whose points are allowed to have non-trivial automorphism groups, e.g. $\mathbb{A}^1/\mathbb{C}^* = \underbrace{(\mathbb{A}^1\setminus 0)/\mathbb{C}^*}_{=\text{point}} \sqcup \underbrace{0/\mathbb{C}^*}_{=\text{point}/\mathbb{C}^*}$

One of the easiest example of stacks, and the only one we will encounter in this course, are quotient stacks $\boxed{S/G}$ where S is a scheme and G is an algebraic group $\curvearrowright S$

Example: if Q is a quiver with vertex set I, and $n=(n_i)_{i\in I}$ is a dimension vector, the stack of n-dimensional representations of Q is

$$\operatorname{Rep}_n Q = \prod_{e=\overrightarrow{ij}} \underbrace{\operatorname{Hom}(\mathbb{A}^{n_i}, \mathbb{A}^{n_j})}_{\cong \mathbb{A}^{n_i\cdot n_j}} \Big/ \prod_{i\in I} GL_{n_i}$$

acts by left/right multiplication

over any field $\mathbb{K}$, a $\mathbb{K}$-valued point of $\operatorname{Rep}_n Q$ is an isomorphism class of n-dimensional representations of Q, in other words, $\operatorname{Rep}_n Q(\mathbb{K}) = \underline{\mathrm{Ob}(\operatorname{Rep}_n Q \text{ over } \mathbb{K})}/\sim$

Therefore
$$\boxed{H_{\operatorname{Rep} Q} = \bigoplus_{n\in\mathbb{N}^I} \left(\text{functions } \operatorname{Rep}_n Q(\mathbb{K}) \longrightarrow \mathbb{C}\right)}$$
where $\mathbb{K}=\mathbb{F}_{\sqrt{q}}$

the Hall multiplication can be interpreted using the stack of extensions

$$\operatorname{Ext}_{n,n'} Q = \left\{\left(\phi_e : \mathbb{A}^{n_i+n_i'} \longrightarrow \mathbb{A}^{n_j+n_j'} \text{ which send } \mathbb{A}^{n_i} \text{ to } \mathbb{A}^{n_j}\right)\right\} \Big/ \begin{array}{l}\{(g_i \in GL_{n_i+n_i'})\}\\ \text{which preserve}\\ \text{this s.e.s.}\end{array}$$

fix s.e.s.
$0 \to \mathbb{A}^{n_i} \to \mathbb{A}^{n_i+n_i'} \to \mathbb{A}^{n_i'} \to 0$

Note the maps

$$\begin{tikzcd}
 & \operatorname{Ext}_{n,n'} Q \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\
\operatorname{Rep}_{n'} Q \times \operatorname{Rep}_n Q & & \operatorname{Rep}_{n+n'} Q
\end{tikzcd}$$

where p_2 remembers the ϕ_e and p_1 remembers $(\phi_e|_{\mathbb{A}^{n_i'}}, \phi_e|_{\mathbb{A}^{n_i}})$

Prop: the Hall multiplication on $H_{\operatorname{Rep} Q}$ is given by $\boxed{p_{2*}\circ p_1^*}$, where for any morphism $\pi: Y \to Z$, we define $\pi_*: (\text{fcts on } Y(\mathbb{K})) \to (\text{fcts on } Z(\mathbb{K}))$
$\pi^*: (\text{fcts on } Z(\mathbb{K})) \to (\text{fcts on } Y(\mathbb{K}))$
as composition with π^{-1} & π, respectively

At this point, we ask ourselves: if instead of "functions on $\mathrm{Rep}\, Q(\mathbb{K})$" we use some other cohomology theory on $\mathrm{Rep}\, Q$ (which is endowed with push-forward and pull-back maps), will we get interesting algebras? Yes!

<u>Motivation</u>: Grothendieck's faisceaux-fonctions correspondence (used in the proof of the Weil conjectures)

let X be a scheme/stack defined over $\mathbb{Z}$

constant sheaves over a locally closed stratification of X

$$\left\{\begin{array}{c}\text{constructible sheaves on } X \\ \text{(e.g. } \mathbb{C})\end{array}\right\} \longrightarrow \left\{\text{functions } X(\mathbb{F}_{\sqrt{q}}) \to \mathbb{C}\right\}$$

$$\mathcal{F} \rightsquigarrow \mathrm{Tr} \text{ of Frobenius on } (\mathcal{F}_x)_{x\in X(\mathbb{F}_{\sqrt{q}})}$$

$\Big\downarrow$ characteristic cycle (uses Morse theory)

$$\left\{\text{algebraic cycles on } T^*X\right\} \xleftarrow{\ ch\ } \left\{\text{algebraic K-theory of } T^*X\right\}$$

- Lusztig made the top-left corner, when $X = \mathrm{Rep}\, Q$, into an associative algebra, and used perverse sheaves to realize $U_q^+(\mathfrak{g})$ inside it
- Grojnowski made the bottom-right corner into an associative algebra, and noted that you can actually find $U_q^+(L\mathfrak{g}) \supset U_q^+(\mathfrak{g})$ inside it

more on this later

In a few words, there is a stack of extensions

$$\begin{array}{ccc} & \text{Some stacks of extensions} & \\ \swarrow^{p_1} & & \searrow^{p_2} \\ T^*\mathrm{Rep}_{n'} Q \times T^*\mathrm{Rep}_n Q & & T^*\mathrm{Rep}_{n+n'} Q \end{array}$$

One defines the K-theoretic Hall algebra of the quiver Q as

$$K\text{-}HA_Q = \bigoplus_{n\in\mathbb{N}^I} K_{\mathbb{C}^*}(T^*\mathrm{Rep}_n Q)$$

made into an algebra by $\boxed{p_{2*}(\mathcal{L} \otimes p_1^*)}$

some line bundle that comes from the difference between algebraic cycles and algebraic K-theory

Before we make this precise, let us talk some more about (equivariant) algebraic K-theory